2.0 THE THEORY OF THE MOTION OF IAPETUS

2.1 INTRODUCTION

The motion of Iapetus, the ninth major satellite of Saturn, is characterised by significant Solar perturbations and by the large inclination of the orbit to the equator plane of the primary. The dominant perturbing forces upon Iapetus are, in decreasing order of magnitude :

- The Sun
- Titan
- The oblateness of Saturn

The theory of Iapetus was developed first by H. Struve (1888) whose model includes periodic Solar perturbations plus secular terms in the node, apse, inclination and eccentricity. Sinclair (1974) revised Struve's theory by adding perturbations due to Titan and the oblateness of Saturn, plus extra Solar terms. Sinclair's aim was to include all terms greater than $0^\circ.001$ in Saturnicentric position, which corresponds to $0''.01$ seen from Earth at 8.5AU, in order to make effective use of post-1967 photographic observations of the positions of the satellites of Saturn.

Rapaport (1978) suggested that Iapetus is affected by a close commensurability of its mean motion with that of Titan. The mean motions are very nearly in the ratio 5:1 so that $5n_I - n_T$ is $0^\circ.113$ per day. Rapaport added several terms to Sinclair's theory containing the angle $5\lambda_I - \lambda_T$ in the argument. He also added a Solar term to the eccentricity and made a new determination of the mean motion of Iapetus.

The most recent theory of Iapetus is by Harper et al (in submission). We have added several significant Solar terms, notably in the node, and we have evaluated the overall size of the 5:1 Titan perturbations, calculating them in a different manner to Rapaport and finding them to be far smaller than Rapaport suggests.

2.2 USE OF THE NUMERICAL INTEGRATION AS A REFERENCE MODEL

In the current work, Sinclair's (1974) theory of Iapetus was chosen as a basis for further development since it is the most recent full theory and it was constructed using standard techniques involving classical orbital elements. It was thus a relatively easy task to add extra perturbation terms without re-casting the entire theory. Indeed, Sinclair's theory is an extension of that of H. Struve.

Following Sinclair and Taylor (1985), the analytic theory of Iapetus was compared to elements derived from a numerical integration of the motions of Titan, Hyperion and Iapetus over a period of 50 years. The numerical integration has several properties which make it ideal as a reference model of this kind :

1. Osculating elements can be obtained from the integration at regularly spaced dates over a long period. By contrast, observational data are irregularly scattered over the period 1870 to 1983, occurring in small clusters around each opposition. Data are totally absent between 1930 and 1967.

   In addition, it is impossible to extract information about osculating elements directly from observations whereas a numerical integration provides elements with little trouble via instantaneous position and velocity vectors.

2. A numerical integration which has been fitted to observations (Sinclair and Taylor (1985)) is a dynamically consistent representation of the real satellite system. The force model of the integration is not a truncated approximation (as is the case with the disturbing function of an analytical theory) and thus the integration implicitly contains all periodic perturbations limited only by the accuracy of the coordinates produced by the integration calculations.

Comparison of the elements from the analytical theories with those obtained from the integration indicates periodic terms which may have been omitted from the theories. Knowledge of the periods of these terms and their approximate amplitudes enables them to be identified in the expansion of the disturbing function.

The numerical integration used in this work is that of Sinclair and Taylor (1985). Its force model includes Solar perturbations, the second and fourth harmonics of the gravity field of Saturn, and mutual satellite perturbations including those due to Rhea (though Rhea's effect upon Iapetus is negligible and is effectively a small augmentation of Saturn's J_2 coefficient). The numerical integration method is described in detail in chapter 5.

2.3 DISCUSSION OF RESIDUALS FROM SINCLAIR'S THEORY

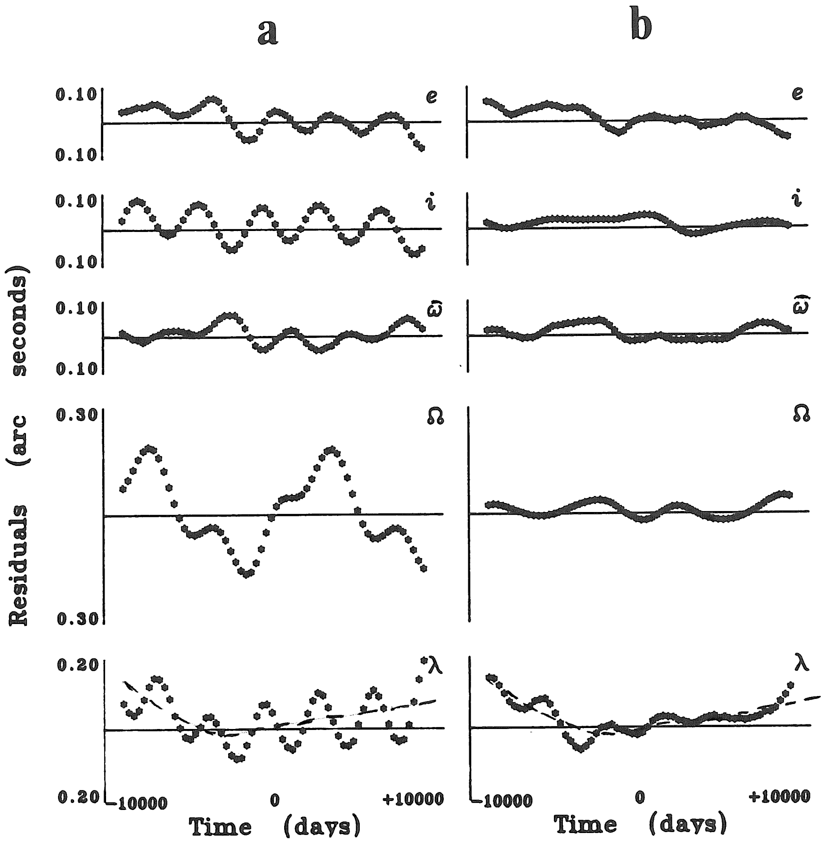

Figure 1. Integration-minus-theory residuals

In Figure 1 we show graphs of the residuals in the eccentricity, apse, inclination, node and mean longitude of Iapetus. The elements are referred to the mean Equator and Equinox of B1950 and the residuals are in the sense Integration-minus-Theory. Thus they represent the terms to be added to the theory in order to make it agree with the integration. The residuals are formed by taking the difference Integration - Theory at intervals of 15 days and calculating average values every 300 days from groups of 20 points. By this method we may eliminate terms of short period (less than 150 days) to enable the long-period behaviour of the elements to be seen more clearly.

The residuals are scaled so that the graphs show the effect upon the observed positions of the satellite at a mean opposition distance of 8.5 AU. In particular, this means that the graphs labelled 'Apse' and 'Node' are plots of $e\Delta\varpi$ and $\sin i\ \Delta\Omega$ respectively and hence they can be compared directly with the graphs of Δe and Δi. Column (a) shows the residuals between the integration and the theory of Sinclair (1974) whilst column (b) shows the residuals between the integration and the improved theory developed in this chapter.

All the elements in column (a) show residuals which are of long period. The period of the residuals in the eccentricity, inclination and apse is 3500 days while that of the node is around 10000 days. Recalling that the mean orbital period of Saturn is 10759 days, we may immediately identify these residuals as Solar perturbations. The terms we seek in the Solar disturbing function have arguments which contain the mean longitude of the Sun but not that of Iapetus. The derivation of these Solar terms is given in the next section.

The most noticeable feature of the perturbations in the mean longitude is a periodic term with a period of approximately 3000 days. It is superimposed upon a term which is secular or of very long period (indicated with a dotted line). The periodic term closely matches the period of the 5:1 Titan perturbations discussed by Rapaport. This is investigated further in a subsequent section, where we shall show that the introduction of 5:1 Titan perturbations reduces the residuals in the mean longitude.

2.4 SOLAR PERTURBATIONS UPON IAPETUS

As a first step in calculating the perturbations upon Iapetus due to the Sun, we must develop an expansion for the disturbing function of the Sun. This is given by the following expression (for the derivation, refer to appendix C).

[1] $$R_s = GM_s\left(\frac{1}{\Delta_s} - \frac{\underline{r}.\underline{r}_s}{r_s^3}\right)$$

where G = gravitational constant

M_s = mass of the Sun

Δ_s = the distance between the Sun and Iapetus

$\underline{r}$ = the Saturnicentric position vector of Iapetus

$\underline{r}_s$ = the Saturnicentric position vector of the Sun

$r_s = |\underline{r}_s|$

We may write this as

[2] $$R_s = GM_s/r_s \left\{(1 + (r/r_s)^2 - 2\,(r/r_s)\cos X)^{-1/2} - (r/r_s)\cos X\right\}$$

$$= GM_s/r_s \sum_{p=1}^{\infty} (r/r_s)^p \, P_p(\cos X)$$

where $P_p(x)$ is the Legendre polynomial of order p.

Now r/r_s is a small quantity of order 0.0025, so we need only take the first term. We write

[3] $$R_s = GM_s/r_s \,(r/r_s)^2 \,(-\tfrac{1}{2} + \tfrac{3}{2}\cos^2 X).$$

By Kepler's third law, we have

[4] $$n_s^2 a_s^3 = GM_s$$

and we may re-write the expression for R_s as

[5] $$R_s = n_s^2 a^2 (r/a)^2 (a_s/r_s)^3 (-\tfrac{1}{2} + \tfrac{3}{2}\cos^2 X).$$

We must now express R_s in terms of the orbital elements of Iapetus and the Sun. The expansions of powers of the radii vectores are straightforward and may be found in Brouwer and Clemence (1961). In order to evaluate $\cos^2 X$ we consider the orbit planes of Iapetus and the Sun referred to the ecliptic and equinox of B1950.0 as in the accompanying figure.

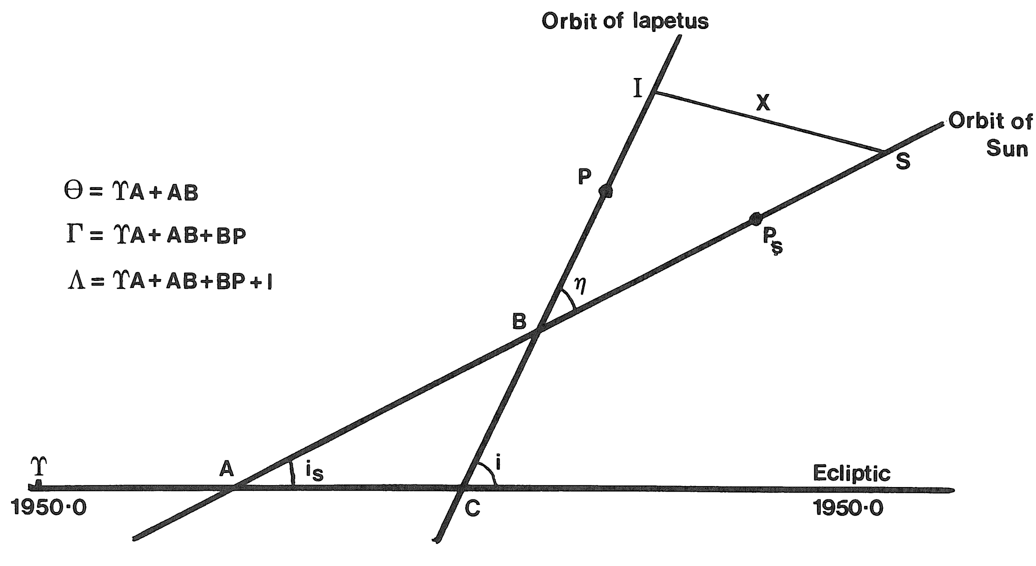

Figure 2. The orbits of Iapetus and the Sun

I and S denote Iapetus and the Sun respectively and P and P_s are the pericentres of Iapetus and the Sun. The notation is as follows.

$$g = BP \qquad g_s = BP_s$$
$$f = PI \qquad f_s = P_s S$$
$$\vartheta = CB \qquad \Theta = \Omega_s + AB$$
$$\Gamma = \Omega_s + AB + BP = \Theta + BP$$
$$\omega = \Omega + CP \qquad \omega_s = \Omega_s + AP_s$$

From the formulae of spherical trigonometry we have

[6] $$\cos X = \cos(g + f) \cos(g_s + f_s) + \cos \eta \sin(g + f) \sin(g_s + f_s)$$

where f = the true anomaly of Iapetus

g = the argument of the apse of Iapetus i.e. the arc of the orbit from the ascending node upon the reference plane to the apse

f_s, g_s are the corresponding quantities for the Sun

η = the inclination of the orbit of Iapetus to that of the Sun.

We may rewrite this in terms of the mean anomalies and eccentricities by substituting the equation of the centre for both bodies (see for example Brouwer and Clemence (1961)). Substitution of the elliptic expansions and evaluation of the resulting series for R is a lengthy process and so the computer algebra package CAMAL-F was employed. CAMAL-F is designed to handle the Fourier series of celestial mechanics and it includes facilities for making substitutions of the form

[7] $$\vartheta = \psi + \text{a Fourier series}$$

of which the equation of the centre is an example. The CAMAL-F program for expansion of the Solar disturbing function is given in Appendix A.

We seek those terms in the disturbing function which contain the mean anomaly of the Sun but not that of Iapetus. In addition to the terms given by Sinclair (1974) we find six significant terms.

[8] $$R_1 = (9/8)n_s^2\ a^2\ e_s^2\ (1 - 6\gamma^2 + 6\ \gamma^4)\ \cos 2\ell_s$$

[9] $$R_2 = (105/16)n_s^2\ a^2\ e_s\ e^2\ (1 - 2\gamma^2)\ \cos(3\ell_s + 2\ g_s - 2g)$$

[10] $$R_3 = (51/4)n_s^2\ a^2\ e_s^2\ \gamma^2\ (1 - \gamma^2)\ \cos(4\ell_s + 2g_s)$$

[11] $$R_4 = (21/4)n_s^2\ a^2\ e_s\ \gamma^2\ (1 - \gamma^2)\ \cos(3\ell_s + 2g_s)$$

[12] $$R_5 = -(3/4)n_s^2\ a^2\ e_s\ \gamma^2(1 - \gamma^2)\ \cos(\ell_s + 2g_s)$$

[13] $$R_6 = (3/4)n_s^2\ a^2\ e_s\ (1 - 6\ \gamma^2 + 6\ \gamma^4)\ \cos\ \ell_s$$

where we have written $\gamma = \sin(\eta/2)$

In order to simplify the calculation of the perturbations, we use elements of the orbit of Iapetus referred to the orbit plane of the Sun about Saturn. We define elements Λ, Γ and Θ corresponding to mean longitude, apse and node as in Figure 2 on page 13. The Lagrange planetary equations become

[14] $$\frac{de}{dt} = -\frac{1}{na^2}\frac{\sqrt{1-e^2}}{e}\frac{\partial R}{\partial \Gamma}$$

[15] $$\frac{d\Gamma}{dt} = \frac{1}{na^2}\left\{\frac{\sqrt{1-e^2}}{e}\frac{\partial R}{\partial e} + \frac{\gamma}{2}\frac{\partial R}{\partial \gamma}\right\}$$

[16] $$\frac{d\Lambda}{dt} = n - \frac{2}{na}\frac{\partial R}{\partial a} + \frac{e}{2na^2}\frac{\partial R}{\partial e} + \frac{\tan\eta/2}{na^2}\frac{\partial R}{\partial \eta}$$

[17] $$\frac{d\eta}{dt} = -\frac{1}{na^2}\frac{1}{2\gamma\sqrt{1-\gamma^2}}\frac{\partial R}{\partial \Theta}$$

[18] $$\frac{d\Theta}{dt} = \frac{1}{na^2}\frac{1}{4\gamma}\frac{\partial R}{\partial \gamma}$$

where only the lowest power of the eccentricity has been retained. We express the arguments of the disturbing function as functions of Λ, Γ and Θ using

[19] $$\ell = \Lambda - \Gamma \qquad \ell_s = \lambda_s - \varpi_s$$
$$g = \Gamma - \Theta \qquad g_s = \varpi_s - \Theta$$

and we calculate the derivatives of the elements substituting each term of the disturbing function into the planetary equations. We assume all the elements on the right hand side of the equations to be constants, except for the mean longitudes which are assumed to vary at constant rates. Upon integration, we obtain the perturbations in e, Γ, η and Θ which we denote by Δe, $\Delta\Gamma$, $\Delta\eta$, $\Delta\Omega$

The form of the perturbation in the eccentricity is independent of the choice of reference plane and so Δe may be quoted directly. However, in order to obtain expressions for the perturbations in the node, apse and inclination with respect to the ecliptic and equinox of B1950 we must apply transformations to $\Delta\Gamma$, $\Delta\eta$ and $\Delta\Omega$. Consider the spherical triangle ABC in Figure 2 on page 13. We may write

$$
[20] \qquad
\begin{aligned}
\cos i &= \cos i_s \cos \eta - \sin i_s \sin \eta \cos (\Theta-\Omega_s) \\
\sin i \cos (\Omega-\Omega_s) &= \sin i_s \cos \eta + \cos i_s \sin \eta \cos (\Theta-\Omega_s) \\
\sin i \sin (\Omega-\Omega_s) &= \sin \eta \sin (\Theta-\Omega_s)
\end{aligned}
$$

from which may be obtained the following derivatives (see appendix B) :

$$
[21] \qquad
\begin{aligned}
\partial i/\partial\eta &= + \cos \vartheta \quad ; & \partial i/\partial\Theta &= - \sin \vartheta \sin \eta \\
\sin i \, \partial\Omega/\partial\eta &= + \sin \vartheta \quad ; & \sin i \, \partial\Omega/\partial\Theta &= + \cos \vartheta \sin \eta \\
\sin i \, \partial\vartheta/\partial\eta &= - \sin \vartheta \cos i \quad ; & \sin i \, \partial\vartheta/\partial\Theta &= + \sin i_s \cos (\Omega-\Omega_s)
\end{aligned}
$$

and hence to first order we may write

$$
[22] \qquad
\begin{aligned}
\Delta i &= \cos \vartheta \, \Delta\eta - \sin \vartheta \sin \eta \, \Delta\Theta \\
\sin i \, \Delta\Omega &= \sin \vartheta \, \Delta\eta + \cos \vartheta \sin \eta \, \Delta\Theta \\
\sin i \, \Delta\vartheta &= - \sin \vartheta \cos i \, \Delta\eta + \sin i_s \cos (\Omega-\Omega_s) \, \Delta\Theta .
\end{aligned}
$$

We note also that

$$[23] \qquad \varpi = \Gamma - \Theta + \Omega + \vartheta$$

therefore

$$[24] \qquad \Delta\varpi = \Delta\Gamma - \Delta\Theta + \Delta\Omega + \Delta\vartheta .$$

From Figure 1 on page 10, we expect the dominant perturbation in the eccentricity and apse to have a period which is one third that of the Sun. Consequently, its argument must contain the angle $3\ell_s$. We notice also from the Lagrange planetary equation for de/dt that in order for a term to contribute to Δe, its argument must contain Γ, and hence in the original expression for R_s, it must contain g, the apse of Iapetus. The only term satisfying these requirements is R_2. From this term we find

$$[25] \qquad \Delta e = (35n_s/8n)\, e\, e_s \sqrt{1 - e^2}\, (1 - 2\gamma^2) \cos(3\ell_s + 2g_s - 2g)$$

$$[26] \qquad e\, \Delta\varpi = (35n_s/8n)\, e\, e_s \sqrt{1 - e^2}\, (1 - 2\gamma^2) \sin(3\ell_s + 2g_s - 2g).$$

We now calculate the perturbations in the node and inclination. From equation [22] we see that Δi and $\sin i\ \Delta\Omega$ can be expressed as a combination of $\Delta\eta$ and $\Delta\Theta$. Consider a term in the disturbing function with argument Ψ. Its contribution to $\Delta\eta$ may be written

$$[27] \qquad \Delta\eta = A \cos \Psi$$

and $\Delta\Theta$ may be written

$$[28] \qquad \Delta\Theta = B \sin \Psi$$

where A and B are functions of n, n_s, e, e_s and γ obtained from the Lagrange planetary equations. They may be treated as constants. Substituting into equation [22] we obtain

$$[29] \qquad \begin{aligned} \Delta i &= \tfrac{1}{2}(A + B \sin \eta) \cos (\Psi + \vartheta) + \tfrac{1}{2}(A - B \sin \eta) \cos (\Psi - \vartheta) \\ \sin i\ \Delta\Omega &= \tfrac{1}{2}(A + B \sin \eta) \cos (\Psi + \vartheta) + \tfrac{1}{2}(A - B \sin \eta) \cos (\Psi - \vartheta). \end{aligned}$$

As an example we consider the term R_4

$$R_4 = 21/4 \quad n_s{}^2 a^2 e_s \gamma^2 (1 - \gamma^2) \cos (3\ell_s + 2g_s).$$

We substitute equations [19] into the argument of the term :

[30] $\Psi = 3\ell_s + 2g_s \;=\; 3\lambda_s - \omega_s - 2\Theta$

then

[31] $d\Theta/dt = 21/8 \quad (n_s{}^2/n)\; e_s \;(1 - 2\gamma^2) \cos (3\ell_s + 2g_s)$

giving

[32] $\Delta\Theta = 7/8 \quad (n_s/n)\; e_s \;(1 - 2\gamma^2) \sin (3\ell_s + 2g_s)$

and

[33] $d\eta/dt = -21/4 \quad (n_s{}^2/n)\; e_s \;(1 - \tfrac{1}{2}\gamma^2) \sin (3\ell_s + 2g_s)$

giving

[34] $\Delta\eta = 7/8 \quad (n_s/n)\; e_s \;(1 - \tfrac{1}{2}\gamma^2) \cos (3\ell_s + 2g_s)$

so we may write

[35]
$$A = 7/4 \quad (n_s/n)\; e_s\gamma(1 - \tfrac{1}{2}\gamma^2)$$
$$B = 7/8 \quad (n_s/n)\; e_s\gamma(1 - 2\gamma^2).$$

The terms R_1, R_3, R_5 and R_6 are treated in exactly the same way. We present the coefficients A and B for each of the terms in tabular form below.

Term	Argument (Ψ)	A	B
R_1	$2\ell_s$	0	$-(27n_s/16n)\ e^2_s$
R_3	$4\ell_s + 2g_s$	$(51n_s/16n)\ \gamma\ e^2_s$	$(51n_s/32n)\ e^2_s$
R_4	$3\ell_s + 2g_s$	$(7n_s/4n)\ e_s\ \gamma(1 - \frac{1}{2}\gamma^2)$	$(7n_s/8n)\ e_s\ (1 - 2\gamma^2)$
R_5	$\ell_s + 2g_s$	$-(3n_s/4n)\ e_s\ \gamma(1 - \frac{1}{2}\gamma^2)$	$-(3n_s/8n)\ e_s\ (1 - 2\gamma^2)$
R_6	ℓ_s	0	$-(9n_s/4n)\ e_s$

We adopt the following values for the elements of the orbits of the Sun and Iapetus. The elements are referred to the mean Ecliptic and Equinox of B1950.0. and are for the epoch JD 2409786.0 (1885.67). These elements change very slowly and the coefficients of the terms are rather small so we do not introduce significant errors by adopting fixed values for the nodes, inclinations and eccentricities. For the Sun

$n_s = 0^\circ.0334597$ /day $\qquad e_s = 0.05560$

$i_s = 2^\circ.4909$ $\qquad \Omega_s = 113^\circ.158$

and for Iapetus

$n = 4^\circ.53795711$ /day $\qquad e_o = 0.028796$

$i_o = 18^\circ.4606$ $\qquad \Omega_o = 143^\circ.1209$

From the spherical triangle ABC in Figure 2 on page 13 we may write

$$\begin{aligned} \sin\eta \sin\vartheta &= \sin i_s \sin(\Omega_o - \Omega_s) \\ \sin\eta \cos\vartheta &= \cos i_s \sin i_o - \sin i_s \cos i_o \cos(\Omega_o - \Omega_s) \qquad [36] \\ \cos\eta &= \cos i_s \cos i_o + \sin i_s \sin i_o \cos(\Omega_o - \Omega_s) \end{aligned}$$

and we obtain $\eta = 16^\circ.348$, $\gamma = 0.14218$, $\vartheta = 4^\circ.423$.

Since ϑ is a small angle, we may neglect it when calculating the coefficients of the perturbations in i and Ω. The coefficients themselves are rather small and the effect of assuming $\vartheta = 0$ will be negligible. It has the advantage of reducing the number of terms to be added to the expressions for the perturbations in i and Ω, since instead of two terms with arguments $\Psi + \vartheta$ and $\Psi - \vartheta$ we will have only one term with argument Ψ.

We present below the terms to be added to Sinclair's (1974) theory as a result of this work.

[37] $\Delta e = +0.00000496 \cos (3\ell_s + 2g_s - 2g)$

[38] $e\,\Delta\omega = +0.00000496 \sin (3\ell_s + 2g_s - 2g)$ radians

[39] $\Delta i = +0^\circ.0005 \cos (4\ell_s + 2g_s) + 0^\circ.0058 \cos (3\ell_s + 2g_s) - 0^\circ.0024 \cos (\ell_s + 2g_s)$

[40] $\sin i\,\Delta\Omega = -0^\circ.0006 \sin 2\ell_s + 0^\circ.0003 \sin (4\ell_s + 2g_s) + 0^\circ.0028 \sin (3\ell_s + 2g_s) - 0^\circ.0012 \sin (\ell_s + 2g_s) - 0^\circ.0142 \sin \ell_s$

2.5 THE 5:1 QUASI-RESONANCE DUE TO TITAN

Comparison of Sinclair's (1974) theory of Iapetus with the numerical integration reveals a periodic residual in the mean longitude with an amplitude of 0".1 arc-seconds and a period of 3000 days. This may be identified with a 5:1 quasi-resonance due to Titan which was first noted by Plana (1826). It arises from the fact that the mean motion of Titan is very nearly five times that of Iapetus.

[41] $5n_I - n_T = 0^\circ.113$ per day

Thus

[42] $\nu = n_I/(5n_I - n_T) = 40.2$

which suggests that terms which include $5\lambda_I - \lambda_T$ in their argument may give rise to significant perturbations in the mean longitude since the coefficients of such terms include the square of this factor.

This was developed by Rapaport (1978) who investigated a number of these terms using a method attributed to J.L. Simon. Rapaport concluded that such terms may have Saturnicentric amplitudes up to 90 arc-seconds which corresponds to 0".25 as seen from the Earth at 8.5 AU. The residual plot of the mean longitude in Figure 1 on page 10 (column (a)) shows an amplitude of only 0".1 as seen from 8.5 AU and this leads us to suspect that Rapaport's work overestimates the net size of the quasi-resonance terms and gives a somewhat misleading impression of their importance.

We set out to re-calculate these terms using the method employed by Sinclair in his (1974) revision of the theory of Iapetus.

2.5.1 THE DISTURBING FUNCTION

We seek all terms in the disturbing function of Titan upon Iapetus which include $5\ell - \ell_T$ in the argument. (ℓ denotes the mean anomaly of Iapetus and ℓ_T the mean anomaly of Titan). We use the expansion of the planetary disturbing function developed by Newcomb in volume 5 of the "Astronomical Papers for the Use of the American Ephemeris". Using Newcomb's notation, we find that the following sets of indices give suitable terms :

(1)	k = 2	j = 0	j' = 0	i = 3
(2)	k = 1	j = 2	j' = 0	i = 4
(3)	k = 1	j = 1	j' = 1	i = 3
(4)	k = 1	j = 0	j' = 2	i = 2

whence

[43] $$R_1=(3/8)n^2a^2m_T\ \gamma^4(\alpha^2 b_{5/2}^{(3)} - \frac{5}{2}\ \gamma^2\alpha^3(b_{7/2}^{(2)} + b_{7/2}^{(4)}))\ \cos(5\ell-\ell_T+5g_1-g_T)$$

[44] $$R_2=(1/16)n^2a^2m_T\ e_T^2\ \gamma^2\ \alpha(31 + 12\alpha D_\alpha + \alpha^2 D_\alpha^2)\ b_{5/2}^{(4)}\ \cos(5\ell-\ell_T+5g_1-3g_T)$$

[45] $$R_3= -((1/8)n^2a^2m_T\ e\ e_T\ \gamma^2\ \alpha(50 + 16\alpha D_\alpha + \alpha^2 D_\alpha^2)\ b_{5/2}^{(3)}\ \cos(5\ell-\ell_T+4g_1-2g_T)$$

[46] $$R_4=(1/16)n^2a^2m_T\ e^2\ \gamma^2\ \alpha(85 + 20\alpha D_\alpha + \alpha^2 D_\alpha^2)\ b_{3/2}^{(2)}\ \cos(5\ell-\ell_T+3g_1-g_T)$$

where γ = sine of half the mutual inclination of the orbits of Titan and Iapetus

α = ratio of the semi-major axes of Titan to Iapetus = 0.34303

D_α = the differential operator $d/d\alpha$

$b_s^{(i)}$ are Laplace coefficients.

These terms have been verified by deriving them independently from Pierce's (1849) expansion of the planetary disturbing function.

2.5.2 CALCULATION OF THE PERTURBATION IN THE MEAN LONGITUDE

We follow the method of Sinclair (1974) and calculate the perturbations of the elements of the orbit of Iapetus referred to the orbit plane of Titan. This introduces an error since the orbit plane of Titan is not fixed but varies slowly due to the influence of the Sun and the oblateness of Saturn. However, the inertial terms may be neglected in this work since they are very small.

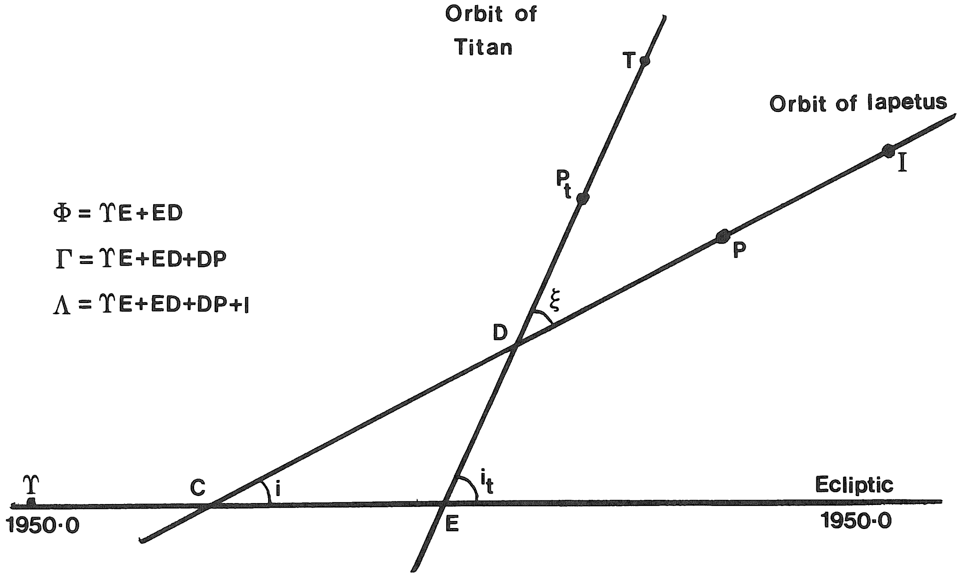

Figure 3. Reference frame for Titan perturbations

The notation is as follows :

$\Omega_T = \Upsilon E$	$\omega_T = \Upsilon E + EP_T$
$\phi = CD$	$\Phi = \Upsilon E + ED$
$g_1 = DP$	$g_T = DP_T$

The relevant Lagrange planetary equations are

[47] $$\frac{da}{dt} = \frac{2}{na}\frac{\partial R}{\partial \Lambda}$$

[48] $$\frac{d\Lambda}{dt} = n - \frac{2}{na}\frac{\partial R}{\partial a} + \frac{e}{2na^2}\frac{\partial R}{\partial e} + \frac{\tan \xi/2}{na^2}\frac{\partial R}{\partial \xi}$$

[49] $$\frac{d\Theta}{dt} = \frac{1}{na^2 \sin\xi} \frac{\partial R}{\partial \xi}$$

[50] $$\frac{d\xi}{dt} = \frac{\tan \xi/2}{na^2} \frac{\partial R}{\partial \Lambda} - \frac{1}{na^2 \sin\xi} \frac{\partial R}{\partial \Theta}$$

and the perturbation in the mean longitude referred to the Ecliptic and Equinox of B1950 may be written

[51] $$\Delta\lambda = \Delta\Lambda - \Delta\Theta + \Delta\Omega + \Delta\phi$$

where (cf. Solar perturbations)

[52] $$\sin i \ (\Delta\Omega + \Delta\phi) = \sin \phi \ (1 - \cos i) \ \Delta\xi + (\sin \xi \cos \phi + \sin i_T \cos (\Omega - \Omega_T))\Delta\Theta.$$

We note that the mean motion n in the equation for $d\Lambda/dt$ must include the perturbations from Δa : since $n^2 a^3$ is a constant we have

[53] $$\Delta n/n = -3/2 \quad \Delta a/a$$

or

[54] $$dn/dt = -3/a^2 \quad \partial R/\partial\Lambda.$$

Thus the first part of $\Delta\Lambda$ contains the double integral

[55] $$\iint \frac{d\Lambda}{dt}\,dt^2 = -\frac{3}{a^2}\iint \frac{\partial R}{\partial \Lambda}\,dt^2$$

If we denote the mean rate of change of the argument of a given term in R by κ then the process of integrating $\partial R/\partial\Lambda$ twice with respect to time will introduce a factor $1/\kappa^2$. In the case of the 5:1 terms under consideration, κ is a small quantity and hence $1/\kappa^2$ will be large. We expect the double integration of dn/dt to yield the most significant part of $\Delta\Lambda$ for such terms.

In the next section we derive the perturbation in λ from a general 5:1 term in the disturbing function.

2.5.3 DERIVATION OF $\Delta\lambda$ FROM ANY 5:1 TERM

Consider any of the 5:1 terms in R given in section 2.5.1; we may write it as

[56] $$R = n^2 a^2 \mu_T \beta \sin (5\ell - \ell_T + jg - j_T g_T)$$

where β is a dimensionless function of e, e_T, γ and α

j , j_T are integers.

We may split $\Delta\lambda$ (equation [51]) into four parts :

1. A term arising from the double integration of dn/dt

2. A term arising from the double integration of $d\Lambda/dt - n$

3. The term $-\Delta\Theta$ in equation [51] plus the contribution of $\Delta\Theta$ to $\Delta\Omega+\Delta\phi$ (equation [52])

4. The contribution of $\Delta\xi$ to $\Delta\Omega+\Delta\phi$.

We may write accordingly

$$[57] \qquad \Delta\lambda = \Delta(\lambda^{(1)} + \lambda^{(2)} + \lambda^{(3)} + \lambda^{(4)}).$$

It is instructive and convenient to treat each part separately. We first make the substitutions

$$\ell = \Lambda - \Gamma \qquad g = \Gamma - \Theta$$

$$\ell_T = \lambda_T - \widehat{\omega}_T \qquad g_T = \widehat{\omega}_T - \Theta$$

into the argument of the term, which becomes

$$5\Lambda + (j - 5)\Gamma + (j_T - j)\Theta - \lambda_T + (1 - j_T)\widehat{\omega}_T.$$

Inspection of the Lagrange planetary equations shows that we shall only be concerned with the coefficients of Λ and Θ since we are not required to evaluate the derivatives of the argument with respect to any of the other angles. We note, therefore, that the coefficient of Λ is 5 and that of Θ is $j_T - j$, and we may resume writing the argument in terms of ℓ, ℓ_T, g and g_T. For brevity, we denote the argument of each term simply as $\Psi = 5\ell - \ell_T + jg - j_T g_T$ and we denote the important ratio $n/(5n - n_T)$ by ν. The value of ν is approximately 40.

Term (1)

The mean motion is dependent upon the semi-major axis by virtue of Kepler's third law, which implies that n^2a^3 is constant for each satellite. Thus a perturbation in the semi-major axis requires a balancing perturbation in the mean motion. Thus we have

[58] $$2\ \Delta n/n = -3\ \Delta a/a$$

from which we may obtain

[59] $$d^2\lambda^{(1)}/dt^2 = dn/dt = -3/a^2\ \partial R/\partial \Lambda$$

which upon integration twice yields

[60] $$\Delta\lambda^{(1)} = -15\ \nu^2 \mu_T \beta \sin \Psi$$

where, as we have said, Ψ is the argument of the term. This part of $\Delta\lambda$ will later be shown to be the largest since it contains the factor ν^2 which augments the other factors in the coefficient by three orders of magnitude.

Term (2)

The remainder of $d\Lambda/dt$ after removing n (which we have dealt with as term (1)) gives us $d\lambda^{(2)}/dt$:

[61] $$\frac{d\lambda^{(2)}}{dt} = -\frac{2}{na}\frac{\partial R}{\partial a} + \frac{e}{2na^2}\frac{\partial R}{\partial e} + \frac{\gamma}{2na^2}\frac{\partial R}{\partial \gamma}$$

where $\gamma = \sin \xi/2$ replaces ξ as the inclination parameter. In order to evaluate $\partial R/\partial a$ it is more convenient to write R as

[62] $$R = \frac{k^2\mu_T\ \beta\ \sin\Psi}{a}$$

using Kepler's third law to write $k^2(M + m_T) = n^2a^3$ but neglecting m_T since it is very small in relation to M, the mass of Saturn.

Then R depends upon the semi-major axis directly, and indirectly via β which is a function of $\alpha = a_T/a$. Clearly,

[63] $$\frac{\partial R}{\partial a} = k^2\mu_T\frac{\partial}{\partial a}\left(\frac{\beta}{a}\right)\sin\Psi$$

and

[64] $$\frac{\partial}{\partial a}\left(\frac{\beta}{a}\right) = -\frac{\beta}{a^2} + \frac{1}{a}\frac{\partial\beta}{\partial a} = -\frac{\beta}{a^2} - \frac{\alpha}{a^2}\frac{\partial\beta}{\partial\alpha}$$

$$= -\frac{1}{a^2}\left(\beta + \alpha\frac{\partial\beta}{\partial\alpha}\right)$$

The derivatives of R with respect to e and γ are straightforward since the dependence is solely via β. Hence we have

$$[65] \quad \frac{d\lambda^{(2)}}{dt} = n\mu_T \left\{ 2\left(\beta + \alpha\frac{\partial\beta}{\partial\alpha}\right) + \frac{e}{2}\frac{\partial\beta}{\partial e} + \frac{\gamma}{2}\frac{\partial\beta}{\partial\gamma} \right\} \cos \Psi$$

and thus

$$[66] \quad \Delta\lambda^{(2)} = \nu\mu_T \left\{ 2\left(\beta + \alpha\frac{\partial\beta}{\partial\alpha}\right) + \frac{e}{2}\frac{\partial\beta}{\partial e} + \frac{\gamma}{2}\frac{\partial\beta}{\partial\gamma} \right\} \sin \Psi$$

Term (3)

We may combine the $-\Delta\Theta$ in equation [51] with the part of $\Delta\Omega+\Delta\phi$ which contains $\Delta\Theta$. Denote this by $\lambda^{\langle 3 \rangle}$. Then

[67]

$$d\lambda^{\langle 3 \rangle} = n\mu_T \; (-1 + (\sin \xi \cos \phi + \sin i_T \cos (\Omega - \Omega_T))/\sin i)\frac{1}{4\gamma}\frac{\partial\beta}{\partial\gamma} \cos \Psi$$

and thus

[68]

$$\Delta\lambda^{\langle 3 \rangle} = \nu\mu_T \; (-1 + (\sin \xi \cos \phi + \sin i_T \cos (\Omega - \Omega_T))/\sin i)\frac{1}{4\gamma}\frac{\partial\beta}{\partial\gamma} \sin \Psi.$$

Term (4)

The remainder of $\Delta\Omega+\Delta\phi$ has a factor $\Delta\xi$. Denoting this by $\Delta\lambda^{\langle 4 \rangle}$ we have

[69] $\Delta\lambda^{(4)} = \sin\phi\ (1 - \cos i)/\sin i\ \ \Delta\xi.$

Thus

[70] $$\frac{d\lambda^{(4)}}{dt} = -n\mu_T \frac{\sin\phi\,(1-\cos i)}{\sin i}\,\beta \sin\Psi$$

and hence

[71] $$\Delta\lambda^{(4)} = \nu\mu_T \frac{\sin\phi\,(1-\cos i)}{\sin i}\,\beta \cos\Psi$$

2.5.4 NUMERICAL VALUES OF THE COEFFICIENTS

We shall use the following numerical values of the constants to evaluate the coefficients of the 5:1 terms.

$\Omega_t - \Omega = 26^\circ.370$ $\qquad$ $i_t = 27^\circ.659$

$\alpha = 0.34303$ $\qquad$ $e_t = 0.0288$

$\mu_t = 2.412\ 10^{-4}$ $\qquad$ $\psi = 36^\circ.1322$

Hence

$\xi = 13^\circ.6964$ $\qquad$ $\phi = 60^\circ.5514$

The values of the nodes and inclinations vary over long periods of time, but it is sufficient to adopt fixed values since the variations in the coefficients will be very small.

The Laplace coefficients are evaluated from their power series expansions. We present the contributions from each of the four parts of $\Delta\lambda$ in the table below.

Term	R_1	R_2	R_3	R_4
Argument $-(5\ell-\ell_T)$	$5g - g_T$	$5g - 3g_T$	$4g - 2g_T$	$3g - g_T$
	Coefficient of sine (argument)			
$\Delta\lambda^{\langle 1\rangle}$	-10.13	- 2.96	+17.29	-25.96
$\Delta\lambda^{\langle 2\rangle}$	+ 0.46	+ 0.04	- 2.13	+ 0.49
$\Delta\lambda^{\langle 3\rangle}$	+ 0.75	+ 0.12	- 0.67	+ 1.01
SUM	- 8.92	- 2.80	-14.49	-24.46
	Coefficient of cosine (argument)			
$\Delta\lambda^{\langle 4\rangle}$	- 0.04	- 0.01	+ 0.03	- 0.05

As expected, the most significant part of the coefficient of sine (argument) comes from $\Delta\lambda^{\langle 1\rangle}$. The coefficients of cosine (argument) from $\Delta\lambda^{\langle 4\rangle}$ are entirely negligible and may be ignored. Indeed, if the derivation of these terms had been carried out rigorously i.e. with respect to a fixed reference frame rather than the orbit plane of Titan, then there would be no cosine terms at all in the perturbation in the mean longitude.

We may now write in full the perturbation in the mean longitude due to the 5:1 quasi-resonance with Titan.

[72] $$\begin{aligned}\Delta\lambda = & -0^\circ.00248 \sin (5\ell - \ell_T + 5g_1 - g_T) \\ & -0^\circ.00078 \sin (5\ell - \ell_T + 5g_1 - 3g_T) \\ & +0^\circ.00403 \sin (5\ell - \ell_T + 4g_1 - 2g_T) \\ & -0^\circ.00679 \sin (5\ell - \ell_T + 3g_1 - g_T)\end{aligned}$$

2.5.5 COMPARISON WITH RAPAPORT'S COEFFICIENTS

Rapaport (1978) carried out his derivation of the 5:1 terms with respect to the equator and equinox of B1950. In his notation we have

ℓ	mean anomaly of Titan
ℓ'	mean anomaly of Iapetus
g ,h	argument of the apse and longitude of the node of Titan
g' ,h'	argument of the apse and longitude of the node of Iapetus

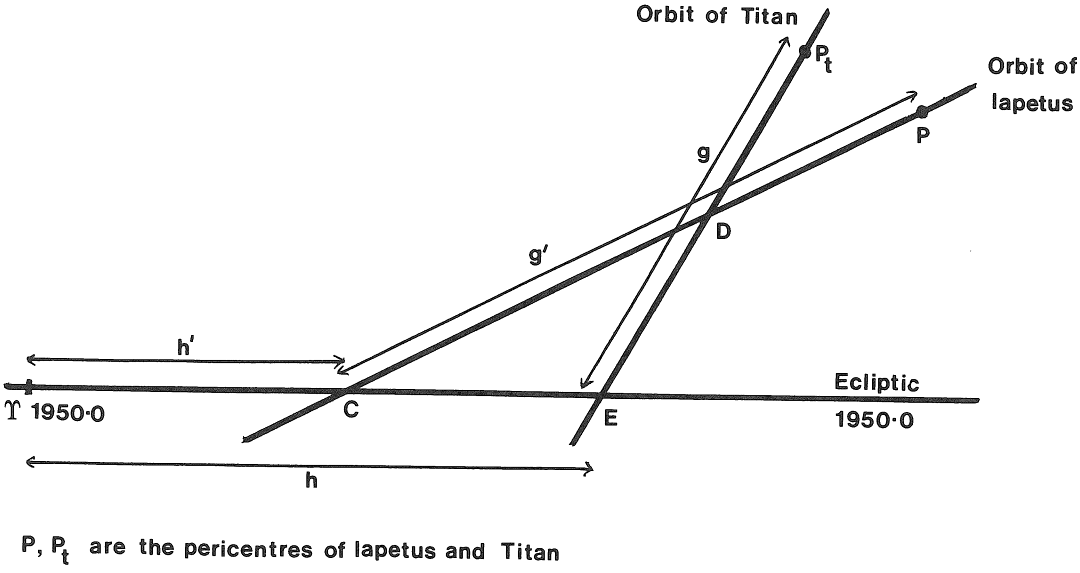

Figure 4. The orbits of Titan and Iapetus

In Rapaport's notation

$$g = EP_T \qquad g' = CP$$

$$h = ♈E \qquad h' = ♈C.$$

In Sinclair's notation

$$g_T = DP_T \qquad g_1 = DP$$

$$\phi = CD \qquad \psi = ED.$$

We may relate Rapaport's notation to that of Sinclair by writing

Rapaport	$\rightarrow$	Sinclair
ℓ	$\rightarrow$	ℓ_T
ℓ'	$\rightarrow$	ℓ
g	$\rightarrow$	$\phi + g_T$
g'	$\rightarrow$	$\psi + g_1$
h	$\rightarrow$	Ω_T
h'	$\rightarrow$	Ω

In this context, the symbol $\rightarrow$ is to be read as 'is replaced by'. Using these substitutions we may transform the arguments of Rapaport's 5:1 terms into a form equivalent to our development. For example, in Rapaport (1978) Table 2 we find the term

$$-42''.64 \sin (5\ell - 5\ell' + g - 5g' + h - h').$$

The argument of this term becomes, upon substitution,

$$\ell_T - 5\ell + \psi + g_T - 5(\phi+g_1) + \Omega_T - \Omega$$

$$= \quad \ell_T - 5\ell + g_T - 5g_1 + (\psi - 5\phi + \Omega_T - \Omega).$$

The first part of this argument varies quite rapidly and may be recognised as the argument of the term in equation [43] and of the first term of equation [72], though with the opposite sign due to Rapaport's notation. The second part may be regarded as a slowly-varying quantity since it contains only angles depending upon the nodes and inclinations of the orbits, which vary over periods of the order of several hundred to several thousand years.

Carrying out this transformation upon the terms given by Rapaport in his Table 2, we notice that the first five terms share the same rapidly-varying part and they correspond to the term in equation [43]. Likewise, Rapaport's terms 6 and 7 have a rapidly-varying part

$$\ell_T - 5\ell + g_T - 3g_1$$

and terms 8, 9 and 10 have a rapidly-varying part

$$\ell_T - 5\ell + 2g_T - 4g_1.$$

Thus they correspond to the terms given in equations [46] and [45].

Consider the first five terms of Rapaport (1978) Table 2. We may write them as

[73] $$\sum_{i=1}^{5} c_i \sin (\ell_T - 5\ell + g_T - 5g_1 + \Psi_i)$$

where c_i is the amplitude given by Rapaport and Ψ_i is the slowly-varying part of the argument, a linear combination of ϕ, ψ and $\Omega_T - \Omega$. Now write

[74] $$C_i = c_i \sin \Psi_i \quad , \quad S_i = c_i \cos \Psi_i .$$

Then [73] may be written as

[75] $$\sum_{i=1}^{5}\left(S_i \sin(\ell_T - 5\ell + g_T - 5g_1) + C_i \cos(\ell_T - 5\ell + g_T - 5g_1)\right)$$

$$= \left(\sum_{i=1}^{5} S_i\right) \sin(\ell_T - 5\ell + g_T - 5g_1) + \left(\sum_{i=1}^{5} C_i\right) \cos(\ell_T - 5\ell + g_T - 5g_1).$$

We may evaluate the coefficients C_i and S_i using values of ϕ, ψ and $\Omega_T - \Omega$ derived from Sinclair (1977) :

$$\Omega = 142^\circ.574 \qquad \iota = 18^\circ.3206$$

$$\Omega_T = 168^\circ.944 \qquad \iota_T = 27^\circ.659$$

and hence

$$\phi = 60^\circ.551 \qquad \psi = 36^\circ.132$$

where all values are for the epoch JD 2415020 and are referred to the ecliptic and equinox of 1950.

We find that Rapaport's terms reduce to

$$+ 12''.28 \sin(\ell_T - 5\ell + g_T - 5g_1) + 3''.98 \cos(\ell_T - 5\ell + g_T - 5g_1).$$

Repeating this operation upon Rapaport's other 5:1 terms yields

$$+ 20''.90 \sin(\ell_T - 5\ell + g_T - 3g_1) - 7''.33 \cos(\ell_T - 5\ell + g_T - 3g_1)$$

and

$$- 15''.99 \sin(\ell_T - 5\ell + 2g_T - 4g_1) - 9''.44 \cos(\ell_T - 5\ell + 2g_T - 4g_1).$$

The following table gives a comparison of Rapaport's 5:1 terms with those derived in this work.

Argument $-(5\ell - \ell_T)$	Our coefft. of sin(arg)	Rapaport's coeffts. sin(arg)	cos(arg)	No. of terms in Rapaport
$5g - g_T$	$- 8''.92$	$-12''.28$	$+3''.98$	5
$5g - 3g_T$	$- 2''.80$			0
$4g - 2g_T$	$+14''.49$	$+15''.99$	$-9''.44$	3
$3g - g_T$	$-24''.46$	$-20''.90$	$-7''.33$	2

There is good agreement in the magnitudes of the coefficients when they are written in this form. We note that when Rapaport's terms are combined and written in a physically more meaningful way, their amplitudes are far smaller than Rapaport's published coefficients (1978, table 2) would suggest.

2.6 COMPARISON OF THE THEORIES WITH THE NUMERICAL INTEGRATION.

We now compare the theories of Iapetus developed in this chapter with the numerical integration of Sinclair and Taylor (1985). We may identify four variants of the theory of Iapetus :

1. Sinclair (1974).

2. Harper et al (1) : Sinclair (1974) plus solar terms in eccentricity, apse, node and inclination developed in this chapter.

3. Harper et al (2) : Harper et al (1) plus the 5:1 Titan terms in the mean longitude developed in this chapter.

4. Harper-Rapaport : Harper et al (1) plus the 5:1 Titan terms developed by Rapaport (1978).

Each of these theories is fitted to the numerical integration by a process of repeated corrections to the elements of the orbit of Iapetus. Thus the comparison shows how closely the chosen theory represents the reference integration, rather than how closely it represents the real orbit of Iapetus. However, the integration itself has been obtained by fitting to photographic observations over a period of some 15 years. Moreover, as explained in the introduction to this chapter, the integration may be regarded as a dynamically consistent representation of a satellite system which closely resembles the real system over a limited span of time. We may expect that a comparison of the various theories with this integration will provide an indication of the precision of such theories when we eventually test them against observational data.

The process of fitting the theories to the integration yields the residual differences between the position of Iapetus given by the integration and the position given by the chosen theory. If we write

$\underline{r}_i$ = position of Iapetus at any time given by the numerical integration

$\underline{r}_t$ = position of Iapetus at the same time given by the theory

then the residual of greatest interest in the comparison is

$$s = | \underline{r}_i - \underline{r}_t | .$$

As part of the fitting process, we obtain values of s at 1216 regularly-spaced dates across the 50-year span of the integration. We form the root-mean-square of these values and we note the maximum value for each theory once the fitting process has converged. In the table below we give the values of the RMS and maximum residual for each of the theories.

Theory	Root Mean Square Residual (A.U.)	Maximum Residual (A.U.)
Sinclair	0.00000543	0.00002023
Harper et al (1)	0.00000401	0.00001586
Harper et al (2)	0.00000348	0.00001311
Harper/Rapaport	0.00000382	0.00001387

Comparison of the values for Sinclair and for Harper et al (1) shows the significant improvement made by the addition of the Solar perturbations. Both the RMS and maximum residuals are reduced by a quarter. This is principally due to the term in the node

$$\sin \iota \, \Delta\Omega \;=\; -0^{\circ}.0142 \sin \ell_s .$$

The maximum effect of such a term upon the Saturnicentric position of Iapetus is 0.0000 0590 AU. The reduction in the maximum residual is of this order and we may attribute the greater part of this reduction to the main term in the node.

The effect of this long-period term in reducing the RMS residual is much smaller since the contribution of the term to $\Delta\Omega$ varies in size as $|\sin \ell_s|$ varies over the period of the term. During a third of the period of this term, for example, $|\sin \ell_s|$ is less than a half. We should note that the interval over which the comparison is being made (50 years) is only a little more than one and a half periods of the term and so we may not expect its contribution to be properly reflected in the RMS residual. Nonetheless, the significance of this term and the other Solar terms is clear from the 25% reduction in the RMS residuals. This corresponds to approximately 0".034 as seen from 8.5 AU ; the reduction in the maximum residual corresponds to 0".106 at 8.5 AU.

We may also see the improvement in the theory by considering the graphs of the averaged residuals of the osculating elements plotted as a function of time. In Figure 1 on page 10 we show the residuals from Sinclair's (1974) theory in column (a) and those from "Harper et al (2)" (i.e. Sinclair (1974) plus Solar terms in node, inclination, apse and eccentricity plus 5:1 terms in the mean longitude derived by the method of Sinclair) in column (b). The significant periodic residual in the node has been removed, as have the periodic residuals in inclination, eccentricity and apse.

Now we consider the improvement to the theory due to the addition of the 5:1 quasi-resonance terms in the mean longitude. The residual graphs of the elements show that the periodic residuals in $\delta\lambda$ have been almost eliminated by the addition of the 5:1 terms developed in this chapter.

Comparison of the RMS and maximum theory-minus-integration residuals also shows some interesting results. The improvement in the fit of the theory to the integration may be seen by comparing the RMS residuals of Harper et al (1) i.e. without 5:1 terms, and Harper et al (2) which include these terms. The RMS residual has been reduced by 53 parts in 401 or approximately 13%. This is about half as large as the improvement produced by the addition of the Solar terms and corresponds to 0".013 as seen from 8.5 AU. Again, the reduction in the maximum residual is larger in absolute terms, though it is of the same relative size, some 17% of the maximum residual from Harper et al (1). This reduction corresponds to 0".067 seen from 8.5 AU.

The overall reduction in RMS and maximum residuals produced by addition of the Solar terms and the 5:1 Titan terms may be summarised as follows.

	RMS	Maximum
Absolute reduction	0.00000195 AU	0.00000712 AU
Relative reduction	35.9%	35.2%
Equivalent arc at 8.5 AU	0".047	0".173

The terms developed in this chapter significantly improve the closeness of the fit between the analytical theory of Iapetus and the motion of the satellite as given by Sinclair and Taylor's integration. Both RMS and maximum residuals are reduced by one third.

Sinclair and Taylor fitted Sinclair's (1974) theory to photographic observations of the satellites made between 1967 and 1982 and obtained a root-mean-square residual in the Titan-Iapetus data of 0".22. The new theory of Iapetus has an RMS residual nearly 0".05 smaller than Sinclair's theory when compared to the integration. This is a significant fraction of the 0".22 residual of Sinclair and Taylor and we may expect a similar improvement when the new theory is compared to the same photographic data.

Comparison of the residuals given by the Harper-Rapaport theory with those for Harper et al (1) shows the effect of adding Rapaport's 5:1 terms. The reduction in the RMS residual is less than half that produced by adding the 5:1 terms developed in this chapter, and the reduction in the maximum residual is only 3/4 as large. Rapaport's development of the 5:1 terms is incomplete as it only includes 3 of the 4 terms which have been included in this work. However, Rapaport's (1978) paper omits much detail such as analytic expressions for the coefficients in his table 2 and the values of the constants used to evaluate them. As a consequence, further critical analysis of his 5:1 terms in comparison with the derivation herein cannot be made. It is sufficient to note that the 5:1 terms developed in this chapter are to be preferred to Rapaport's terms in the form in which Rapaport presents them.